

# LIFTING AUTOMORPHISMS ON ABELIAN VARIETIES AS DERIVED AUTOEQUIVALENCES

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**ABSTRACT.** We show that on an Abelian variety over an algebraically closed field of positive characteristic, the obstruction to lifting an automorphism to a field of characteristic zero as a morphism vanishes if and only if it vanishes for lifting it as a derived autoequivalence. We also compare the deformation space of these two types of deformations.

**Keywords.** Abelian Varieties, Automorphisms, Positive characteristic, Derived autoequivalences, Lifting to characteristic zero.

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## INTRODUCTION

Investigations into derived categories of coherent sheaves on a smooth projective varieties as a geometric invariant started with the seminal papers of Mukai and Bondal-Orlov. (For basics see Huybrechts' book [11].) In [4], Bondal-Orlov showed that smooth projective varieties with ample canonical or anti-canonical bundle are determined up to isomorphism by their derived categories. On the other hand, Mukai [15] showed that this was not the case for Abelian Varieties by explicitly constructing non-isomorphic derived equivalent Abelian Varieties. Further work by many authors, led to examples of mostly Calabi-Yau varieties (varieties with vanishing first Chern class) that are not-isomorphic but derived equivalent. In case the variety is a curve or a surface (over  $\mathbb{C}$ ) [5] or is an Abelian variety [23], it turned out that the number of non-isomorphic derived equivalent varieties to a chosen variety is finite, although unbounded [10]. However, outside the case of Calabi-Yau varieties, Lesieutre [13] has constructed an example of a rational threefold which is derived equivalent to infinitely many non-isomorphic three-folds. In general, the number of non-isomorphic derived equivalent smooth projective varieties is countable [1]. This approach to understanding geometry via derived categories got a major boost with the homological mirror symmetry conjecture of Kontsevich [12], leading to immense interest in understanding and constructing derived autoequivalence of smooth projective varieties over  $\mathbb{C}$ . On the other hand, the group of derived autoequivalences for smooth projective varieties over fields of characteristic  $p$  is largely unexplored other than a few results for K3 surfaces in [27] and for Abelian Varieties in [23], where it is shown that the group of derived autoequivalences satisfies a short exact sequence.

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Any smooth projective variety  $X$  over an algebraically closed field admits the following three kinds of standard derived autoequivalences via shifts of complexes, automorphism of varieties and twists by line bundles

$$\begin{aligned} \mathbb{Z} &\rightarrow \text{Aut}(D(X)) & n &\mapsto [n] \\ \text{Aut}(X) &\rightarrow \text{Aut}(D(X)) & \sigma &\mapsto \sigma_* = (\sigma^{-1})^* \\ \text{Pic}(X) &\rightarrow \text{Aut}(D(X)) & L &\mapsto M_L, \quad M_L(\mathcal{F}) := \mathcal{F} \otimes L. \end{aligned}$$

All the three morphism are injective group homomorphisms. They sit inside  $\text{Aut}(D(X))$  as the subgroup  $\text{Aut}(X) \times \text{Pic}(X) \times \mathbb{Z}$ .

We would like to compare this subgroup of autoequivalences between smooth projective varieties over fields of characteristic  $p$  which admit a lift to characteristic zero and their lifts. More precisely, let  $X$  be a smooth projective variety over an algebraically closed field  $k$  of characteristic  $p$  such that there exists a smooth projective scheme  $X_V$  over  $V$  a discrete valuation ring which is possibly a finite extension of the ring of Witt vectors  $W(k)$  over  $k$ . The generic fiber of  $X_V$  gives a smooth projective variety over a field  $K$  of characteristic zero. When such an  $X_V$  exists for a variety  $X$ , we say that  $X$  is liftable to characteristic zero or admits a lift to characteristic zero. The obstruction to existence of such a lift is given by  $H^2(X, \mathcal{T}_X)$  [8]. Furthermore, given a lift  $X_V$  of  $X$ , we can try to lift automorphisms of  $X$  or line bundles on  $X$  or derived autoequivalences of  $X$  to  $X_V$ . More precisely, we can ask whether given an automorphism (resp. line bundle, resp. derived autoequivalence) on  $X$  there exists an automorphism (resp. line bundle, resp. derived autoequivalence) on  $X_V$  such that on restriction to the special fiber  $X$  we get back the automorphism (resp. line bundle, resp. derived autoequivalence) we started with. In case a lift of the automorphism (resp. line bundle, resp. derived autoequivalence) exists on  $X_V$ , we can base change to the generic fiber and this gives us a way to compare the automorphism group (resp. Picard group, resp. derived autoequivalence group) of a smooth projective variety over a field of characteristic  $p$  and the generic fiber of a lift of it, which is a smooth projective variety over a field of characteristic zero. Even when the lift of the smooth projective exists, lifting an automorphism (resp. line bundle, resp. derived autoequivalence) can be obstructed. The obstruction for lifting an automorphism (resp. line bundle) on  $X$  to some  $X_V$  is given by  $H^1(X, \mathcal{N}_\Gamma)$  (resp.  $H^2(X, \mathcal{O}_X)$ ), where  $\mathcal{N}_\Gamma$  is the normal sheaf of the graph of the automorphism, see section 2 for more details. There exists lot of examples of smooth projective varieties, starting from a surface which do not lift see references in [8, Chapter 22] or that the variety lifts but automorphisms (resp. line bundles) do not lift to some lift or to none of the lifts, see for example [22], [6].

In case of smooth projective varieties  $X, Y$  over a field, Orlov [24] proved that every derived equivalence is given by a Fourier-Mukai transform, i.e, given a derived equivalence  $\varphi : D(X) \xrightarrow{\cong} D(Y)$ , there exist a complex, say  $P \in D(X \times Y)$  giving a functor of derived categories  $\Phi_P : D(X) \rightarrow D(Y) : \mathcal{F} \mapsto p_{Y*}(\mathbb{R}p_X^* \mathcal{F} \otimes^{\mathbb{L}} P)$ , where  $p_X$  (resp.  $p_Y$ ) is the projection from  $X \times Y$  to  $X$  (resp.  $Y$ ) such that  $\varphi$  is isomorphic to the functor  $\Phi_P$ . The functor  $\Phi_P$  is called a Fourier-Mukai functor and  $P$  is called the Fourier-Mukai kernel of  $\Phi_P$ . It is only determined up to isomorphism in the derived category. In the case of the three standard autoequivalences above the Fourier-Mukai kernels are

$$\begin{aligned} \mathbb{Z} &\rightarrow \text{Aut}(D(X)) & n &\mapsto [n] & \mathcal{O}_\Delta[n] \\ \text{Aut}(X) &\rightarrow \text{Aut}(D(X)) & \sigma &\mapsto \sigma_* = (\sigma^{-1})^* & \mathcal{O}_{\Gamma_\sigma} \\ \text{Pic}(X) &\rightarrow \text{Aut}(D(X)) & L &\mapsto M_L, & M_L(\mathcal{F}) := \mathcal{F} \otimes L \quad \Delta_* L, \end{aligned}$$

where  $\Delta$  is the diagonal embedding of  $X$  in  $X \times X$  and  $\Gamma_\sigma$  is the graph of  $\sigma$  embedded into  $X \times X$ . We remark that the above result of Orlov is not known for the case smooth projective schemes over a DVR or any other base other than

a field, so we cannot conclude that every derived autoequivalence of a lift  $X_V$  is of Fourier-Mukai type. We note that lifting a Fourier-Mukai transform is equivalent to lifting the Fourier-Mukai kernel on  $X \times Y$  to some  $X_V \times Y_W$ . However, it is not known whether one can construct lifts of a Fourier-Mukai transform which on  $D(X_V)$  are not of Fourier-Mukai type. Here, we are going to restrict to deforming a derived autoequivalence of  $X$  as a Fourier-Mukai transform, and thus deform the Fourier-Mukai kernel. Note that lifting an automorphism on  $X$  or the corresponding Fourier-Mukai kernel of the derived autoequivalence on  $X \times X$  in the derived category  $D(X \times X)$  are different deformation problems. And we would like to compare these deformation problems.

This paper answers in negative the following question: can one construct new derived autoequivalences in characteristic zero using non-liftable automorphisms from characteristic  $p$  (Cf., [27, Theorem 3.4])? In the sense that maybe one could lift the derived autoequivalence associated to a non-liftable automorphism. Even though, as it turns out, this is not the case, we show that in case of Abelian varieties, the derived autoequivalence associated to an automorphism has more lifts than just as a morphism.

The main idea of the result below is that when we lift an automorphism of  $X$  as an autoequivalence, we basically lift, as a coherent sheaf, the structure sheaf of the graph of the automorphism over the product  $X \times X$ . The support of this lifted sheaf will give a deformation of the graph of the automorphism and since the deformation is flat, it has to come from a lift of the automorphism itself.

The results and arguments presented here most probably will be well known to experts, but since we did not find them written down in literature, we note them down here.

We break the paper into 2 main sections. In section 1 we recall the basics of Abelian varieties that we would need for understanding their deformation theory, derived autoequivalences and the dimensions of the necessary Hodge groups and deformation spaces. Moreover, we observe that  $p$ -rank of an Abelian variety is a derived invariant. Thus, every Abelian variety derived equivalent to an ordinary Abelian variety is ordinary.

In section 2, we use the deformation-obstruction sequence for the corresponding deformation functors as morphism, or as Fourier-Mukai transform and compare the obstruction and deformation space dimensions.

The main result (Proposition 15, Theorem 11) can be stated as follows:

**Theorem 1.** *An automorphism  $\sigma$  of an Abelian variety  $X$  of dimension  $g$  lifts as a morphism to  $X_V$  if and only if it lifts as a derived autoequivalence of  $D(X)$  to an autoequivalence of  $D(X_V)$  as a Fourier-Mukai transform. Moreover, in case it lifts, there is a  $g$  dimensional space of extra lifts of the automorphism  $\sigma$  as a Fourier-Mukai transform. These extra lifts are derived auto-equivalences given by composition of the lifted automorphism with twist by the lift of the structure sheaf of the graph.*

Lastly, we note that first part of the above result still holds for any smooth projective variety over an algebraically closed field of characteristic  $p$  that admits a lift to characteristic zero.

*Notation:* For an algebraically closed field  $k$  of positive characteristic, we will denote by  $W(k)$  the ring of Witt vectors over  $k$ , sometimes also abbreviated as  $W$ .  $K$  will denote a characteristic zero field.

## 1. ABELIAN VARIETIES

In this section we discuss basic properties of Abelian varieties paying special attention to elliptic curves, for which we prove the results first before going to the general case. For basics and proofs of some of the following results, we refer the reader to the excellent book by Mumford, [16].

**1.1. Basic Properties and Deformations.** Let  $A$  be a  $g$ -dimensional Abelian variety over  $k$ , an algebraically closed field.

The **Hodge groups**  $H^q(A, \Omega_A^p)$  are canonically isomorphic to  $\wedge^p[H^0(A, \Omega_A^1)] \otimes \wedge^q[H^1(A, \mathcal{O}_A)]$ , and  $\dim H^1(A, \mathcal{O}_A) = \dim H^0(A, \Omega_A) = g$ .

**Lifting Abelian Varieties:** Every Abelian variety defined over an algebraically closed field of positive characteristic lifts to characteristic zero, see [18].

The formal deformation space of the lifting of an Abelian variety is given by  $H^1(A, T_A)$ , see [17, Proposition 6.15]. This has dimension  $g^2$ . Since all formal deformations need not be algebraic, the dimension of the algebraic deformations will be smaller than  $g^2$  for  $g \geq 2$ . However in the case of elliptic curves, note that the algebraization of the lift is automatic because the obstruction to lifting an (ample) line bundle lies in the second cohomology group, which vanishes as we are in dimension 1 and we can use Grothendieck's existence theorem to conclude.

To construct algebraic lifts for higher dimensional Abelian varieties, we lift a polarization on the base Abelian variety along with it rather than lifting an ample line bundle as in the case of elliptic curves, (for a reason why we do this, see remark 14) and then use Grothendieck's Existence theorem to conclude. The dimension of the local moduli space of algebraic (polarized) lifts is  $g(g+1)/2$ , see [19] for details. We remark that even though every Abelian variety can be lifted to characteristic 0, this is true only in the weak sense, i.e, not every Abelian variety would admit a lift over the Witt ring but one would need a ramified extension of the Witt ring to get a lift, see [22, Section 13]. However, ordinary Abelian varieties admit a lift over the Witt ring.

**Lifting Automorphisms as morphisms of Abelian Varieties:** There exist non-liftable automorphisms for Abelian varieties. Thus, the full automorphism group of an Abelian variety need not lift. For an example see [22, Section 14]. The obstruction and deformation space of a particular automorphism have been computed below in section 2.

For details about the structure of automorphism group (or more generally endomorphisms) of Abelian varieties, we refer the reader to [16, Chapter IV]. For the explicit description of the group of automorphisms of an elliptic curve see Silverman, [26, Chapter III, Theorem 10.1].

**1.2. Derived Equivalent Abelian Varieties.** Let  $A$  (resp.  $B$ ) be an Abelian variety and  $\hat{A}$  (resp.  $\hat{B}$ ) the dual Abelian variety, we will first recall the definition of the group,  $U(A \times \hat{A}, B \times \hat{B})$ , of isometric isomorphisms of  $A \times \hat{A}$  to  $B \times \hat{B}$ . Note that any morphism  $f : A \times \hat{A} \rightarrow B \times \hat{B}$  can be written as a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the morphism  $a$  maps  $A$  to  $B$ ,  $b$  maps  $\hat{A}$  to  $B$ ,  $c$  maps  $A$  to  $\hat{B}$  and  $d$  maps  $\hat{A}$  to  $\hat{B}$ . Each morphism  $f$  determines two other morphisms  $\hat{f}$  and  $\tilde{f}$  from  $B \times \hat{B}$  to  $A \times \hat{A}$  whose matrices are

$$\hat{f} = \begin{pmatrix} \hat{d} & \hat{b} \\ \hat{c} & \hat{a} \end{pmatrix}, \text{ and } \tilde{f} = \begin{pmatrix} \hat{d} & -\hat{b} \\ -\hat{c} & \hat{a} \end{pmatrix}.$$

We define  $U(A \times \hat{A}, B \times \hat{B})$  as the subset of all isomorphisms from  $A \times \hat{A} \rightarrow B \times \hat{B}$  such that  $f^{-1} = \tilde{f}$  and such an isomorphism is called *isometric*. In case  $A = B$  we call an isometric isomorphism as an isometric automorphism and denote the group of isometric automorphisms as  $U(A \times \hat{A})$ . The following theorem of Orlov [23] and Polishchuk [25], characterizes derived equivalences of two Abelian varieties using isometric isomorphisms.

**Theorem 2** ([11], Corollary 9.49). *Two Abelian varieties  $A$  and  $B$  over an algebraically closed field  $k$  define equivalent derived categories,  $D^b(A) \cong D^b(B)$  if and only if there exists an isomorphism  $f : A \times \hat{A} \xrightarrow{B} \times \hat{B}$  with  $\tilde{f} = f^{-1}$ :*

$$D^b(A) \cong D^b(B) \iff U(A \times \hat{A}, B \times \hat{B}).$$

In the case of elliptic curves, recall that the dual of an elliptic is isomorphic to the curve itself, hence we have

**Theorem 3** ([2] Theorem 2.2). *Let  $E$  and  $F$  be two elliptic curves over an algebraically closed field  $k$  such that  $\Phi : D^b(E) \cong D^b(F)$  as a  $k$ -linear triangulated categories, then there is an isomorphism of  $k$ -schemes  $E \cong F$ .*

The theorem 2 has an easy corollary, recall that the  $p$ -rank of an Abelian variety over an algebraically closed field  $k$  is the rank  $i$  of the  $p$ -torsion group  $A[p] \cong (\mathbb{Z}/p\mathbb{Z})^i$ , where  $i \in [0, \dim(X)]$ , if  $\text{char } k = p$  otherwise  $i = 2g$  and an Abelian variety over a field of characteristic  $p$  is called *ordinary* if its  $p$ -rank is equal to  $\dim(A)$ .

**Corollary 4.** *The  $p$ -rank of an Abelian variety is a derived invariant.*

*Proof.* This follows from the easy observation that  $\hat{A}[p] = A[p]$  and then using the theorem above as isomorphisms preserve  $p$ -torsion.  $\square$

This also follows from the result of Honigs [9] that derived equivalent Abelian varieties are isogenous. This implies that any Abelian variety derived equivalent to an ordinary Abelian variety is ordinary.

**1.2.1. Derived Autoequivalence Group of Abelian Varieties.** The derived autoequivalence group of an Abelian Variety  $A$  over an algebraically closed field  $k$  satisfies the following short exact sequence ([23], see [11, Proposition 9.55]):

$$(1) \quad 0 \rightarrow \mathbb{Z} \oplus (A \times \hat{A}) \rightarrow \text{Aut}(D^b(A)) \rightarrow U(A \times \hat{A}) \rightarrow 0$$

Here  $U(A \times \hat{A})$  is the group of isometric automorphisms. Explicitly, the kernel is generated by shifts  $[n]$ , translations  $t_{a*}$ , and tensor products  $L \otimes ()$  with  $L \in \text{Pic}^0(A)$ .

## 2. LIFTING AUTOMORPHISMS AS AUTOEQUIVALENCES

In this section we will compare the two ways of lifting an automorphism of an Abelian variety and prove our main theorem. We begin by analyzing the case of elliptic curves and then generalize to higher dimensional Abelian varieties.

Let  $E$  be an elliptic curve with  $j$ -invariant not equal to 0 or 1728 over  $k$  an algebraically closed field of characteristic  $p > 0$ . Let  $\sigma : E \rightarrow E$  be an automorphism of  $E$ . This automorphism induces a Fourier-Mukai equivalence on the derived categories given by the Fourier-Mukai kernel  $\mathcal{O}_{\Gamma(\sigma)}$ , where  $\mathcal{O}_{\Gamma(\sigma)}$  is the push forward of the structure sheaf of the graph of  $\sigma$  to  $X \times X$  considered as a coherent sheaf in  $D^b(E \times E)$ :

$$\Phi_{\mathcal{O}_{\Gamma(\sigma)}} : D^b(E) \xrightarrow{\cong} D^b(E).$$

This reinterpretation of an automorphism as an perfect complex in the derived category provides us with another way of deforming the automorphism. We can now deform it as a perfect complex rather than deforming it just as a morphism. However, since the complex we start with is just a coherent sheaf, the deformations of it as a perfect complex will still be coherent sheaves, see [27, remark after proof of Theorem 3.4]. Thus we can encode the two ways of lifting an automorphism in the following deformation functors.

Let  $R$  be an Artinian local  $W(k)$ -algebra with residue field  $k$ ,  $E_R$  be an infinitesimal deformation of  $E$  over  $R$  and consider the following two deformation functors: first is the **deformation functor of an automorphism as a morphism** given by

$$(2) \quad \begin{aligned} F_{aut} : & (\text{Artin local } W(k)\text{-algebras with residue field } k) \rightarrow (\text{Sets}) \\ & R \mapsto \{\text{Lifts of automorphism } \sigma \text{ to } R, \text{ i.e., pairs } (E_R, \sigma_R)\}, \end{aligned}$$

where by lifting of automorphism  $\sigma$  over  $R$  we mean that there exists an infinitesimal deformation  $E_R$  of  $E$  and an automorphism  $\sigma_R : E_R \rightarrow E_R$  which reduces to  $\sigma$ , i.e., we have the following commutative diagram:

$$\begin{array}{ccc} E_R & \xrightarrow{\sigma_R} & E_R \\ \uparrow & & \uparrow \\ E & \xrightarrow{\sigma} & E. \end{array}$$

The second one is the **deformation functor of an automorphism as a coherent sheaf**, defined as follows:

$$(3) \quad \begin{aligned} F_{coh} : & (\text{Artin local } W(k)\text{-algebras with residue field } k) \rightarrow (\text{Sets}) \\ & R \mapsto \{\text{Deformations of } \mathcal{O}_{\Gamma(\sigma)} \text{ to } R\}/\text{iso}, \end{aligned}$$

where by deformations of  $\mathcal{O}_{\Gamma(\sigma)}$  to  $R$  we mean that there exists an infinitesimal deformation  $Y_R$  of  $Y := E \times E$  over  $R$  and a coherent sheaf  $\mathcal{F}_R$ , which is a deformation of the coherent sheaf  $\mathcal{O}_{\Gamma(\sigma)}$  and  $\mathcal{O}_{\Gamma(\sigma)}$  is considered as a coherent sheaf on  $E \times E$  via the closed embedding  $\Gamma(\sigma) \hookrightarrow E \times E$ . Isomorphisms are defined in the obvious way.

Observe that there is a natural transformation  $\eta : F_{aut} \rightarrow F_{coh}$  given by

$$(4) \quad \begin{aligned} \eta_R : & F_{aut}(R) \longrightarrow F_{coh}(R) \\ (\sigma_R : & E_R \rightarrow E_R) \mapsto \mathcal{O}_{\Gamma(\sigma_R)}/E_R \times E_R. \end{aligned}$$

There is a deformation-obstruction long exact sequence connecting the two functors.

**Proposition 5** ([27] Proposition 3.6). *Let  $i : \Gamma(\sigma) \hookrightarrow E \times E$  be a closed embedding with  $E$  integral and projective scheme of finite type over  $k$ ,  $\sigma : E \rightarrow E$  an automorphism and  $\Gamma(\sigma)$  its graph. Then there exists a long exact sequence*

$$(5) \quad \begin{aligned} 0 \rightarrow H^0(\mathcal{N}_{\Gamma(\sigma)}) \rightarrow \text{Ext}_{E \times E}^1(\mathcal{O}_{\Gamma(\sigma)}, \mathcal{O}_{\Gamma(\sigma)}) \rightarrow H^1(\mathcal{O}_{\Gamma(\sigma)}) \rightarrow \\ H^1(\mathcal{N}_{\Gamma(\sigma)}) \rightarrow \text{Ext}_{E \times E}^2(\mathcal{O}_{\Gamma(\sigma)}, \mathcal{O}_{\Gamma(\sigma)}) \rightarrow \dots, \end{aligned}$$

where  $\mathcal{N}_{\Gamma(\sigma)}$  is the normal bundle of  $\Gamma(\sigma)$ .

**Remark 6.** Note that the obstruction spaces for the functors  $F_{aut}$  and  $F_{coh}$  are  $H^1(\mathcal{N}_{\Gamma(\sigma)})$  and  $\text{Ext}_{E \times E}^2(\mathcal{O}_{\Gamma(\sigma)}, \mathcal{O}_{\Gamma(\sigma)})$  respectively. See, for example, [8, Theorem 6.2, Theorem 7.3]. The same results give us the tangent spaces for the functors  $F_{aut}$  and  $F_{coh}$  and they are  $H^0(\mathcal{N}_{\Gamma(\sigma)})$  and  $\text{Ext}_{E \times E}^1(\mathcal{O}_{\Gamma(\sigma)}, \mathcal{O}_{\Gamma(\sigma)})$ .

**Lemma 7.** *Let  $E$  be an elliptic curve over  $k$  and  $\sigma : E \rightarrow E$  an automorphism of  $E$ , the normal bundle of  $\Gamma(\sigma) : E \rightarrow E \times E$  is trivial.*

*Proof.* Use [7, Proposition II.8.20] for the case  $r = 1$ .  $\square$

Thus, for an elliptic curve  $E$  and an automorphism  $\sigma$  on it, we can compute the obstruction and deformation spaces for  $F_{aut}$  and they are  $T_{F_{aut}} = H^0(E, \mathcal{N}_{\Gamma(\sigma)}) = H^0(E, \mathcal{O}_E) = k$  and  $Obs_{F_{aut}} = H^1(E, \mathcal{N}_{\Gamma(\sigma)}) = H^1(E, \mathcal{O}_E) = k$ .

Next, note that there is a canonical isomorphism  $Ext_{E \times E}^i(\mathcal{O}_{\Gamma(\sigma)}, \mathcal{O}_{\Gamma(\sigma)})$  and  $Ext_{E \times E}^i(\mathcal{O}_E, \mathcal{O}_E)$  given by  $\sigma$ , so to compute the obstruction and deformation spaces for  $F_{coh}$ , we need to compute  $Ext_{E \times E}^i(\mathcal{O}_E, \mathcal{O}_E)$ . Recall that these groups are just the Hochschild cohomology groups of the elliptic curve  $E$  and using Hochschild-Kostant-Rosenberg (HKR) isomorphism [3, Theorem 1.3], we get that

$$\begin{aligned} Ext_{E \times E}^1(\mathcal{O}_E, \mathcal{O}_E) &= H^1(E, \mathcal{O}_E) \oplus H^0(E, \mathcal{T}_E^1) \\ &= H^1(E, \mathcal{O}_E) \oplus H^0(E, \mathcal{O}_E) = k \oplus k, \end{aligned}$$

where the second equality follows as  $\mathcal{T}_E$  is a free sheaf and

$$\begin{aligned} Ext_{E \times E}^2(\mathcal{O}_E, \mathcal{O}_E) &= H^2(E, \mathcal{O}_E) \oplus H^1(E, \mathcal{T}_E) \oplus H^0(E, \mathcal{T}_E^2) \\ &= H^2(E, \mathcal{O}_E) \oplus H^1(E, \mathcal{O}_E) \oplus H^0(E, \wedge^2 \mathcal{O}_E) = H^1(E, \mathcal{O}_E) = k. \end{aligned}$$

Thus dimensions of the  $Ext$  groups are 2 and 1 respectively. With this we have computed the deformation and obstruction spaces for  $F_{coh}$ , that is deformation as a derived autoequivalence, what is left to show is that the lifted structure sheaf of the graph does induce a derived autoequivalence. Indeed, let  $\mathcal{F}_W$  be the lift of the structure sheaf of the graph  $\mathcal{O}_{\Gamma(\sigma)}$  to  $E_W \times E_W$ , where  $E_W$  is a lift of  $E$  over  $W$ . Then using Nakayama's lemma, we note that  $\Delta_* \mathcal{O}_{E_W} \rightarrow \mathcal{F}_W \circ \mathcal{F}_W^\vee$  and  $\mathcal{F}_W^\vee \circ \mathcal{F}_W \rightarrow \Delta_* \mathcal{O}_{E_W}$  are quasi-isomorphism, where  $\mathcal{F}_W \circ \mathcal{F}_W^\vee$  denotes the Fourier-Mukai kernel of the composition  $\Phi_{\mathcal{F}_W} \circ \Phi_{\mathcal{F}_W^\vee}$ , explicitly given by  $p_{13*}(p_{12}^*(\mathcal{F}_W) \otimes p_{23}^*(\mathcal{F}_W^\vee[1]))$ , for the case of elliptic curves. This argument also works for the higher dimensional Abelian varieties case, although the shift in that case is by  $[g]$ , where  $g$  is the dimension of the Abelian variety. Now we are ready to prove first part of the main result in the case of elliptic curves.

**Proposition 8.** *For an elliptic curve  $E$  over an algebraically closed field  $k$  of characteristic  $p > 0$ , any automorphism  $\sigma : E \rightarrow E$  lifts to a lift  $E_K$  of  $E$  over a field  $K$  of characteristic zero, if and only if the Fourier-Mukai transform induced by the structure sheaf of the graph of  $\sigma$ ,  $\mathcal{O}_{\Gamma(\sigma)} \in D^b(E \times E)$ , lifts as an autoequivalence to  $D^b(E_K) \rightarrow D^b(E_K)$ .*

*Proof.* From the long exact sequence 5 above, putting in the computations of the groups from the preceding paragraph we get the exact sequence:

$$0 \rightarrow k \rightarrow k \oplus k \xrightarrow{\alpha} k \xrightarrow{\beta} k \xrightarrow{\gamma} k \rightarrow \dots,$$

where we note that  $\alpha$  has to be a surjection (it cannot be zero, as the exactness would then imply that  $k \oplus k \cong k$ ), thereby making  $\beta$  a zero map and  $\gamma$  an injection. Thus, we get our result.  $\square$

**2.0.2. The case of higher dimensional Abelian varieties.** For Abelian varieties of dimension  $g \geq 2$  just the computations of obstruction-deformation sequence will be insufficient to conclude that every automorphism lifts as a morphism if and only if it lifts as an autoequivalence. However, in this case we observe that support of the lifted Fourier-Mukai kernel sheaf is actually the graph of a lift of an automorphism. This argument works for any smooth projective variety, not just only for Abelian varieties, see remark 12 below.

Let  $A$  be an Abelian variety over  $k$ , an algebraically closed field of positive characteristic and let  $\sigma : A \rightarrow A$  be an automorphism of  $A$ , the definitions of the

previous section can be transported directly over to higher dimensional Abelian varieties from the case of elliptic curves.

**Lemma 9.** *The normal bundle  $\mathcal{N}_{\Gamma(\sigma)}$  of  $\Gamma(\sigma) : A \rightarrow A \times A$  is a free sheaf of rank  $g$  on  $A$ , where  $g$  is the dimension of  $A$ .*

*Proof.* This follows from the following maps of the short exact sequences (definition of normal sheaf, [7, Chapter II.8 Page 182])

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_{A/k} & \longrightarrow & T_{A \times A/k} \otimes \mathcal{O}_A & \longrightarrow & \mathcal{N}_{A/A \times A} \longrightarrow 0 \\ & & \cong \uparrow & & \cong \uparrow & & \cong \uparrow \\ 0 & \longrightarrow & T_{A,e} \otimes \mathcal{O}_A & \longrightarrow & (T_{A \times A,(e,e)} \otimes \mathcal{O}_{A \times A}) \otimes \mathcal{O}_A & \longrightarrow & N_{A,e} \otimes \mathcal{O}_A \longrightarrow 0, \end{array}$$

where the last isomorphism is induced by the first two.  $\square$

Thus, we can compute the tangent and deformation spaces for  $F_{aut}$  as  $H^0(\mathcal{N}_A) = k^g$  and  $H^1(\mathcal{N}_A) = k^{g^2}$ . Next we compute the *Ext* groups and use these computations to show that the exact sequence 5 is not enough anymore to conclude the main result. We will be using HKR isomorphism to compute the *Ext* groups and over fields of positive characteristic this isomorphism only exists when the dimension of the variety is less than the characteristic of the base field, we are forced to put the assumption on characteristic. However, since we are only doing the computations to illustrate the failure of a method, by adding this assumption we do not lose much.

**Lemma 10.** *The groups *Ext* for Abelian variety  $A$  in the exact sequence 5 can be computed as follows in case the characteristic of the base field  $k$  is greater than the dimension of  $A$ :*

$$Ext^1(\mathcal{O}_A, \mathcal{O}_A) = k^{2g} \text{ and } Ext^2(\mathcal{O}_A, \mathcal{O}_A) = k^{2g^2-g}.$$

*Proof.* We will again use HKR isomorphism [3], thus

$$Ext^1(\mathcal{O}_A, \mathcal{O}_A) = HH^1(A) = H^1(A, \mathcal{O}_A) \oplus H^0(A, \mathcal{T}_A) = k^g \oplus k^g$$

and

$$\begin{aligned} Ext^2(\mathcal{O}_A, \mathcal{O}_A) &= HH^2(A) = H^2(A, \mathcal{O}_A) \oplus H^0(A, \mathcal{T}_A^2) \oplus H^1(A, \mathcal{T}_A) \\ &= k^{g(g-1)/2} \oplus k^{g(g-1)/2} \oplus k^{g^2}. \end{aligned}$$

$\square$

Thus the deformation obstruction long exact sequence 5 becomes

$$0 \rightarrow k^g \rightarrow k^{2g} \xrightarrow{\alpha} k^g \rightarrow k^{2g} \xrightarrow{\gamma} k^{2g^2-g} \rightarrow \dots$$

Note that now  $\alpha$  does not have to be surjective to be non-zero. Thus we cannot say that  $\gamma$  is injective. So we need a geometric argument.

**Theorem 11.** *For any Abelian variety  $A$  over an algebraically closed field  $k$  of characteristic  $p > 0$ , any automorphism  $\sigma : A \rightarrow A$  lifts to a lift  $A_K$  of  $E$  over a field  $K$  of characteristic zero, if and only if the Fourier-Mukai transform induced by the structure sheaf of the graph of  $\sigma$ ,  $\mathcal{O}_{\Gamma(\sigma)} \in D^b(A \times A)$ , lifts as an autoequivalence to  $D^b(A_K) \rightarrow D^b(A_K)$ .*

*Proof.* Since deformation of a morphism as an autoequivalence is just the deformation of the structure sheaf of the graph as a coherent sheaf in the derived category, to prove the above statement we need to show that the support of the lifted coherent sheaf actually gives us a lift of the automorphism. This follows easily from



[27, Lemma 3.5] and the fact that given a coherent sheaf  $\mathcal{F}$  on a smooth projective variety, the support of the lifted coherent sheaf  $\mathcal{F}_W$  gives a deformation of the support of  $\mathcal{F}$ .  $\square$

Remark 12. Note that the above argument also works in the case of any smooth projective variety admitting a lifting to characteristic zero.

**2.1. Are there extra lifts of automorphisms as autoequivalences?** First, let us remark that for the case of Abelian varieties, we have the following exact sequence:

Lemma 13. *For any Artin local  $W(k)$ -algebra  $R$  with residue field  $k$ , any Abelian scheme  $A$  over  $\text{Spec}(R) = S$  and for any surjection  $R \twoheadrightarrow R'$  of local Artin  $W(k)$ -algebras such that  $A' = A \otimes_R R'$ , there is an exact sequence:*

$$0 \rightarrow \text{Hom}_S(A, A) \rightarrow \text{Hom}'_S(A', A').$$

*Proof.* This follows from [17, Corollary 6.2] and

$$\begin{array}{ccc} \text{Hom}_S(A, A) & \longrightarrow & \text{Hom}_k(A_0, A_0) \\ \downarrow & \nearrow & \\ \text{Hom}_{S'}(A', A') & & \end{array}$$

$\square$

This implies that  $F_{aut}$  is actually a subfunctor of  $F_{A_0}$ , deformations of  $A_0$  as a scheme, and lift of every automorphism is unique upto lift of the base scheme. On the other hand, one can easily see that this is not true for the deformation functor of coherent sheaves, i.e., given a fixed lift of a base Abelian scheme as a scheme, the lift of a coherent sheaf to the fixed lift will not be unique. It will be a torsor under the  $Ext^1$  group. In the particular case we are working with, note that the number of distinct lifts of the base Abelian variety will be precisely  $g$ -dimensional and the  $Ext^1$  group in our case is  $2g$  dimensional.

Remark 14. This is the reason that for constructing algebraizable lifts of Abelian varieties, we do not just lift an line bundle but instead we choose to lift a polarization, which is actually a morphism of the Abelian variety to its dual variety. Thus using again [17, Corollary 6.2], we get that the deformation functor for polarized Abelian varieties is a subfunctor of deformation functor for Abelian varieties.

Thus for Abelian varieties, the answer to the question posed in the heading is yes. There will be a  $g$ -dimensional space of extra lifts of automorphism as autoequivalences and we have

Proposition 15. *The extra lifts of the automorphism as an autoequivalence are given by composition of the lifted automorphism with twist by the lift of the structure sheaf of the graph.*

Note that a lift of the structure sheaf of the graph of  $\sigma : A \rightarrow A$  will be just a line bundle  $L_W$  on the graph of the lifted automorphism  $\sigma_W : A_W \rightarrow A_W$  which reduces to the structure sheaf of the graph  $\mathcal{O}_{\Gamma(\sigma)}$ .

2.1.1. *Comparison with the deformations of the induced automorphism on product  $A \times \hat{A}$ .* Recall from Theorem 2, that to every derived equivalence  $\Phi_{\mathcal{F}} : D^b(A) \rightarrow D^b(A)$  one can associate an isometric automorphism  $f_{\mathcal{F}}$  of  $A \times \hat{A}$  (in case of elliptic curve  $E$  an automorphism of  $E \times E$ ). This association gives a corresponding transformation on the level of deformation functors, i.e., a natural transformation between the deformation functors  $F_{coh, \mathcal{F}} \rightarrow F_{aut, f_{\mathcal{F}}}$ , where  $\mathcal{F}$  is a (shifted) coherent

sheaf [11, Proposition 9.53]. Note that the association of  $\mathcal{F}$  with  $f_{\mathcal{F}}$  was **not** a 1-1 correspondence, therefore the natural transformation is not injective at the level of tangent spaces of  $F_{coh,\mathcal{F}}$  and  $F_{aut,f_{\mathcal{F}}}$ . This fits well with the discrepancy in the number of lifts of  $\mathcal{F}$  as an sheaf (which is not unique) and  $f_{\mathcal{F}}$ , which is unique for a chosen lift of the base Abelian variety as its deformation functor is a subfunctor for the deformation functor of Abelian varieties as schemes.

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